Nonlinear Perron-Frobenius theorem

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QUANTPI

SIAM LA21 Minitutorial
Applied Nonlinear Perron-Frobenius Theory
May 18, 2021

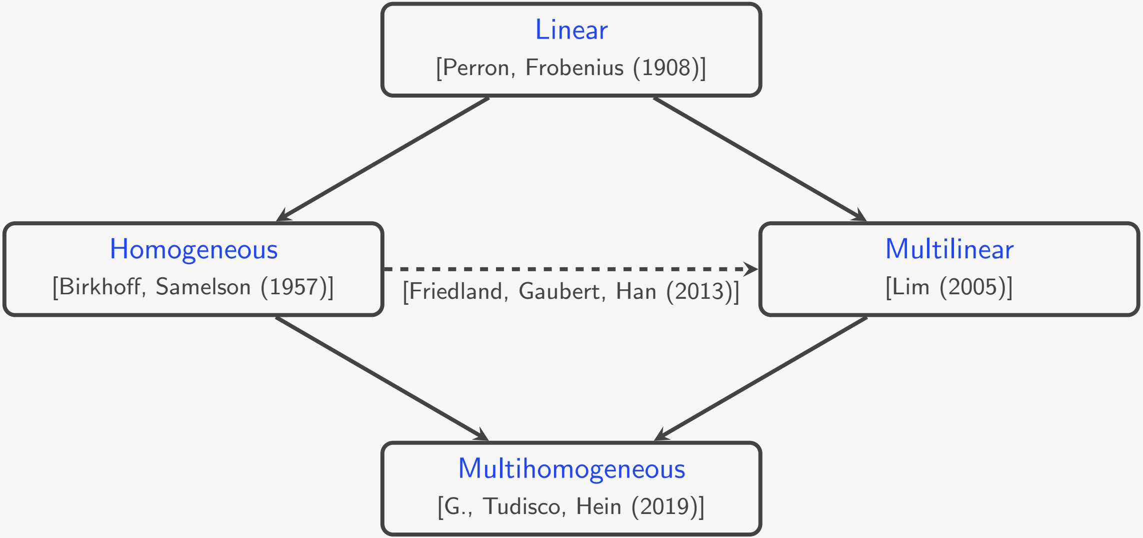
Oskar Perron
1880–1975

Georg Frobenius
1849–1917

The Perron-Frobenius theory is both useful and elegant. It is a testament to the fact that beautiful mathematics eventually tends to be useful, and useful mathematics eventually tends to be beautiful.

C. D. Meyer

Perron-Frobenius theory



Operator norms

$$M \in \mathbb{R}^{m \times n}, \quad T \in \mathbb{R}^{\ell \times m \times n}, \qquad 1 < p, q, r < \infty, \quad \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$$

$$\|M\|_{2,2} \quad = \quad \max_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2}$$

$$\|M\|_{p,q} \quad = \quad \max_{x \neq 0} \frac{\|Mx\|_q}{\|x\|_p}$$

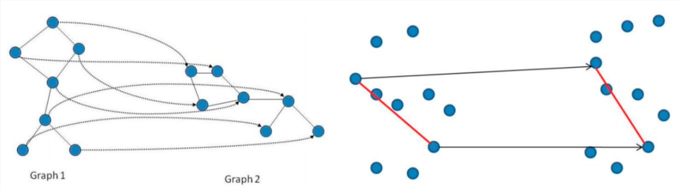
NP-hard
[Steinberg (2005)]

$$\|T\|_{p,q,r} \quad = \quad \max_{x \neq 0} \frac{\sum_{ijk} T_{ijk} x_i y_j z_k}{\|x\|_p \|y\|_q \|z\|_r}$$

NP-hard
[Hillar, Lim (2009)]

Graph matching

[Nguyen, G., Hein (2015)]

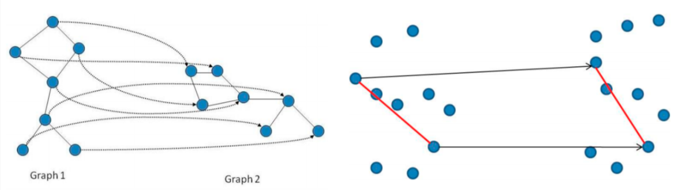


[Duchenne et al. (2011)]

$$\max_{\substack{X\in\{0,1\}^{m\times n}\\ \sum_j X_{i,j}=1,\ \forall i}} \sum_{(i_1 j_1),(i_2 j_2)} H_{(i_1 j_2),(i_2 j_2)} X_{i_1,j_1} X_{i_2,j_2}$$

Graph matching spectral relaxation

[Nguyen, G., Hein (2015)]

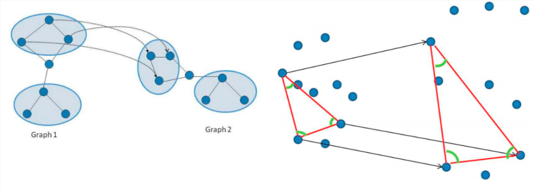


[Duchenne et al. (2011)]

$$\max_{\substack{X\in\mathbb{R}_+^{m\times n}\\ \|X\|_F=\sqrt{m}}} \sum_{(i_1j_1),(i_2j_2)} H_{(i_1j_1),(i_2j_2)} X_{i_1,j_1} X_{i_2,j_2}$$

Hypergraph matching spectral relaxation

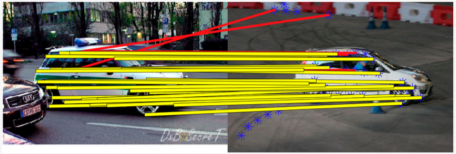
[Nguyen, G., Hein (2015)]



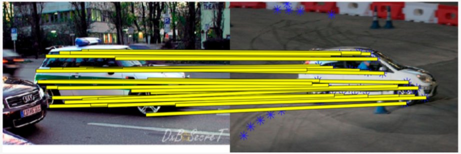
[Duchenne et al. (2011)]

$$\max_{\substack{X \in \mathbb{R}_+^{m \times n} \\ \|X\|_F = \sqrt{m}}} \sum_{(i_1 j_1 k_1),(i_2 j_2 k_2)} H_{(i_1 j_1 k_1),(i_2 j_2 k_2)} X_{i_1,j_1} X_{i_2,j_2} X_{i_3,j_3}$$

Hypergraph matching spectral relaxation



Linear approach



Tensor approach

[Nguyen, G., Hein (2015)]

Linear Perron-Frobenius theorem

$$\|M\|_{2,2} \quad = \quad \max_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2}$$

Nonnegative vectors and component-wise operations

$$\begin{aligned} \mathbb{R}^n_+ &= \{x \in \mathbb{R}^n \colon x_1, \ldots, x_n \geq 0\} \\ \mathbb{R}^n_{++} &= \{x \in \mathbb{R}^n \colon x_1, \ldots, x_n > 0\} \end{aligned}$$

$$\begin{array}{lcll} x \succeq y & \Leftrightarrow & x_i \geq y_i & \forall i \\ x \succ y & \Leftrightarrow & x_i > y_i & \forall i \\ x \succneqq y & \Leftrightarrow & x \succeq y, & x \neq y \end{array}$$

$$x^\theta = (x_1^\theta, \ldots, x_n^\theta) \qquad \forall \theta \in \mathbb{R}, \quad x \succ 0$$

Nonnegative matrices

$$
\begin{aligned}
M \in \mathbb{R}_+^{m\times n} \quad &\Leftrightarrow \quad M_{ij} \geq 0 \quad && \forall i,j \\
&\Leftrightarrow \quad Mx \succeq 0 \quad && \forall x \succeq 0 \\
&\Leftrightarrow \quad Mx \succeq My \quad && \forall x \succeq y
\end{aligned}
$$

Irreducible and primitive matrices

$M \in \mathbb{R}_+^{n \times n}$

M is irreducible if

$$\sum_{k=0}^{n} M^k x \succ 0 \qquad \forall x \gneqq 0$$

M is primitive if

$$\exists k \geq 0 \quad \text{s.t.} \quad M^k x \succ 0 \qquad \forall x \gneqq 0$$

$$\text{primitive} \quad \Rightarrow \quad \text{irreducible}$$

Linear Perron-Frobenius theorem

$M \in \mathbb{R}_+^{n\times n}$

$$Mx = \lambda x, \qquad x \succeq 0, \qquad \|x\| = 1 \qquad (\star)$$

Theorem [Perron, Frobenius (1908)]

M irreducible $\Rightarrow$ Exists unique solution (u, λ) to $(\star)$,

$$u \succ 0 \quad \text{and} \quad \lambda = \rho(M)$$

M primitive $\Rightarrow$ $\forall x^{(0)} \succ 0, \quad x^{(k)} = Mx^{(k-1)}$

$$\lim_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|} = u$$

Examples

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

not irreducible
no positive eigenvector

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

not primitive
no convergence of $x^{(k)}$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

primitive

Application: $\|M\|_{2,2}$

$M \in \mathbb{R}_+^{m \times n}$

$$\|M\|_{2,2} \quad = \quad \max_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2}$$

Corollary

$M^\top M$ irreducible $\quad \Rightarrow \quad$ Exists unique $u \succeq 0$ s.t.

$$\|Mu\|_2 = \|M\|_{2,2}, \qquad \|u\|_2 = 1$$

$M^\top M$ "primitive" $\quad \Rightarrow \quad \forall x^{(0)} \succ 0,$

$$\lim_{k \to \infty} \frac{x^{(k)}}{\|x^{(k)}\|_2} = u, \qquad x^{(k)} = M^\top M x^{(k-1)}$$

Proof sketch: $\|M\|_{2,2}$, $M \in \mathbb{R}_+^{m\times n}$

$\exists$ global maximizer $u \succeq 0$

$$\|M\|_{2,2} = \max_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2} \leq \max_{x\neq 0} \frac{\|\,|Mx|\,\|_2}{\|\,|x|\,\|_2}$$

critical point $\equiv$ eigenvector

$$\nabla \frac{\|Mu\|_2}{\|u\|_2} = 0 \quad \Leftrightarrow \quad M^\top M u = \left(\frac{\|Mu\|_2}{\|u\|_2}\right)^2 u$$

Apply linear Perron-Frobenius theorem

Proof sketch: $\|M\|_{p,p}$, $M \in \mathbb{R}_+^{m\times n}$

$\exists$ global maximizer $u \succeq 0$

$$\|M\|_{p,p} \quad = \quad \max_{x\neq 0} \frac{\|Mx\|_p}{\|x\|_p} \quad \leq \quad \max_{x\neq 0} \frac{\|\,|Mx|\,\|_p}{\|\,|x|\,\|_p}$$

critical point $\equiv$ eigenvector

$$\nabla \frac{\|Mu\|_p}{\|u\|_p} = 0 \qquad \Leftrightarrow \qquad \left(M^\top (Mu)^{p-1}\right)^{\frac{1}{p-1}} = \left(\frac{\|Mu\|_p}{\|u\|_p}\right)^{\frac{p}{p-1}} u$$

Apply homogeneous Perron-Frobenius theorem

Homogeneous Perron-Frobenius theorem

$$\|M\|_{p,q} \quad = \quad \max_{x \neq 0} \frac{\|Mx\|_q}{\|x\|_p}$$

Homogeneous and order-preserving mappings

$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ differentiable

$$
\begin{aligned}
f \ \theta\text{-homogeneous} \quad &\Leftrightarrow \quad f(\alpha x) = \alpha^\theta f(x) \qquad \forall \alpha > 0, x \succeq 0 \\
&\Leftrightarrow \quad D f(x)x = \theta f(x) \qquad \forall x \succeq 0
\end{aligned}
$$

$$
\begin{aligned}
f \ \text{order-preserving} \quad &\Leftrightarrow \quad f(y) \succeq f(x) \qquad \forall y \succeq x \\
&\Leftrightarrow \quad D f(x) \in \mathbb{R}^{n\times n}_+ \qquad \forall x \succeq 0
\end{aligned}
$$

Example: $f(x) = Mx$ with $M \in \mathbb{R}^{n\times n}_+$

Hilbert projective metric

$$d_H(x,y) \quad = \quad \max_{i,j} \ln\left(\frac{x_i}{y_i}\frac{y_j}{x_j}\right) \qquad \forall x,y \succ 0$$

$(\{x \succ 0 \mid \|x\| = 1\}, d_H)$ complete metric space

$$f(x) = \lambda x \qquad \Leftrightarrow \qquad d_H(f(x),x) = 0$$

Lemma [Birkhoff, Samelson (1957)]

$f : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$, θ-homogeneous and order-preserving

$$d_H(f(x),f(y)) \quad \leq \quad \theta\, d_H(x,y) \qquad \forall x,y \succ 0$$

Spectral radius

$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, θ-homogeneous

$$f(x) = \lambda x \qquad \Leftrightarrow \qquad f(\alpha x) = \alpha^{\theta-1}\lambda\,\alpha x, \quad \forall \alpha > 0$$

$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, 1-homogeneous, continuous

$$\rho(f) \qquad = \qquad \lim_{k\to\infty}\left(\sup_{x \gneq 0} \frac{\|f^k(x)\|}{\|x\|}\right)^{1/k}$$

Homogeneous Perron-Frobenius theorem I

$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, continuous, 1-homogeneous, order-preserving

Theorem [Nussbaum, Eveson, Lemmens, ... (1988)]

$\sum_{k=0}^n f^k(x) \succ 0, \quad \forall x \gneq 0,$

$\Rightarrow \qquad \exists u \succ 0$ s.t. $f(u) = \rho(f)u$ and $\rho(f) = \max\{\lambda : \lambda \text{ eigenvalue}\}$

$\sum_{k=0}^n Df(u)^k x \succ 0, \quad \forall x \gneq 0$

$\Rightarrow \qquad u$ is unique eigenvector in $\mathbb{R}^n_{++}$

Irreducible tensors

$T \in \mathbb{R}_+^{n\times n\times n}$, $f : \mathbb{R}_+^n \to \mathbb{R}_+^n$

$$f(x)_i \quad = \quad \Big(\sum_{j,k=1}^{n} T_{ijk} x_j x_k \Big)^{\frac{1}{2}} \qquad \forall x \succeq 0$$

Lemma [Friedland, Gaubert, Han (2013)]

$$\sum_{k=0}^{n} f^k(x) \succ 0, \qquad \forall x \gneq 0 \qquad \Rightarrow \qquad \sum_{k=0}^{n} Df(u)^k x \succ 0, \qquad \forall x \gneq 0$$

Homogeneous Perron-Frobenius theorem II

$f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, continuous, 1-homogeneous, order-preserving,

$$f(u) = \lambda u, \qquad u \succ 0, \qquad \|u\| = 1$$

Theorem [Nussbaum (1988)]

$$Df(u)^k x \succ 0, \quad \forall x \gneqq 0,$$

$$\Rightarrow \qquad \lim_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|} = u, \qquad \forall x^{(0)} \succ 0, \qquad x^{(k)} = f(x^{(k-1)})$$

Application: $\|M\|_{p,p}$

$M \in \mathbb{R}_+^{m\times n}$, $1 < p < \infty$

$$\|M\|_{p,p} \quad = \quad \max_{x\neq 0} \frac{\|Mx\|_p}{\|x\|_p}$$

Corollary [Boyd (1974)]

$M^\top M$ irreducible $\quad\Rightarrow\quad$ Exists unique $u \succeq 0$ s.t.

$$\|Mu\|_p = \|M\|_{p,p}, \qquad \|u\|_p = 1$$

$M^\top M$ "primitive" $\quad\Rightarrow\quad \forall x^{(0)} \succ 0,$

$$\lim_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|_p} = u, \qquad x^{(k)} = \left(M^\top (Mx^{(k-1)})^{p-1}\right)^{\frac{1}{p-1}}$$

Homogeneous Perron-Frobenius theorem III

$f : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$, θ-homogeneous, order-preserving

Theorem [Bushell (1973)]

$0 < \theta < 1 \quad \Rightarrow \quad$ Exists unique $u \succ 0$ s.t.

$$f(u) = \lambda u, \qquad \|u\| = 1$$

and $\quad \forall x^{(0)} \succ 0, \quad x^{(k)} = f(x^{(k-1)})$

$$\lim_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|_p} = u, \qquad d_H(x^{(k)}, u) \quad \leq \quad \theta^k \frac{d_H(x^{(1)}, x^{(0)})}{1-\theta}$$

Application: $\|M\|_{p,q}$

$$M \in \mathbb{R}_+^{m\times n}, \quad M^\top M\,(1,\ldots,1)^\top \succ 0$$

$$\|M\|_{p,q} \quad = \quad \max_{x\neq 0} \frac{\|Mx\|_q}{\|x\|_p}$$

Corollary [G., Hein (2016)]

$1 < q < p < \infty \quad \Rightarrow \quad$ Exists unique $u \succ 0$ s.t.

$$\|Mu\|_q = \|M\|_{p,q}, \qquad \|u\|_p = 1$$

and $\qquad \forall x^{(0)} \succ 0, \quad x^{(k)} = \big(M^\top (Mx^{(k-1)})^{p-1}\big)^{\frac{1}{q-1}}$

$$\lim_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|_p} = u, \qquad \mu(x^{(k)}, u) \leq \Big(\frac{p-1}{q-1}\Big)^k c$$

Multihomogeneous Perron-Frobenius theorem

$$\|T\|_{p,q,r} \quad = \quad \max_{x \neq 0} \frac{\sum_{i,j,k} T_{ijk} x_i y_j z_k}{\|x\|_p \|y\|_q \|z\|_r}$$

Tensor norm

$M \in \mathbb{R}_+^{n\times n}, \quad \langle x, My \rangle = \sum_{i,j=1}^n M_{ij} x_i y_j, \quad q' = \frac{q}{q-1}$

$$\max_{x\neq 0} \frac{\|Mx\|_{q'}}{\|x\|_p} \quad = \quad \max_{x,y\neq 0} \frac{\langle x, My \rangle}{\|x\|_p \|y\|_q}$$

$T \in \mathbb{R}_+^{n\times n\times n}, \quad T(x,y,z) = \sum_{i,j,k=1}^n T_{ijk} x_i y_j z_k$

$$\|T\|_{p,q,r} \quad = \quad \max_{x,y,z\neq 0} \frac{T(x,y,z)}{\|x\|_p \|y\|_q \|z\|_r}$$

Critical points of tensor norm

$$M \in \mathbb{R}_+^{n\times n}, \quad T \in \mathbb{R}_+^{n\times n\times n}, \quad T(x,y,z) = \sum_{i,j,k=1}^n T_{ijk} x_i y_j z_k$$

$$\nabla \frac{\|Mx\|_{q'}}{\|x\|_p} = 0 \quad \Leftrightarrow \quad M^\top (Mx)^{q'-1} = \lambda x^{p-1}$$

$$\nabla \frac{\langle x, My\rangle}{\|x\|_p \|y\|_q} = 0 \quad \Leftrightarrow \quad \begin{cases} M^\top y & = \lambda x^{p-1} \\ Mx & = \lambda y^{q-1} \end{cases}$$

$$\nabla \frac{T(x,y,z)}{\|x\|_p \|y\|_q \|z\|_r} = 0 \quad \Leftrightarrow \quad \begin{cases} \nabla_x T(x,y,z) & = \lambda x^{p-1} \\ \nabla_y T(x,y,z) & = \lambda y^{q-1} \\ \nabla_z T(x,y,z) & = \lambda z^{r-1} \end{cases}$$

Eigenvector equation of tensor norm

$$M \in \mathbb{R}_+^{n\times n}, \quad T \in \mathbb{R}_+^{n\times n\times n}, \quad T(x,y,z) = \sum_{i,j,k=1}^n T_{ijk} x_i y_j z_k$$

$$\nabla \frac{\|Mx\|_{q'}}{\|x\|_p} = 0 \quad \Leftrightarrow \quad (M^\top (Mx)^{q'-1})^{\frac{1}{p-1}} = \lambda^{\frac{1}{p-1}} x$$

$$\nabla \frac{\langle x, My\rangle}{\|x\|_p \|y\|_q} = 0 \quad \Leftrightarrow \quad \begin{cases} (M^\top y)^{\frac{1}{p-1}} & = \lambda^{\frac{1}{p-1}} x \\ (Mx)^{\frac{1}{q-1}} & = \lambda^{\frac{1}{q-1}} y \end{cases}$$

$$\nabla \frac{T(\cdot, y, z)}{\|x\|_p \|y\|_q \|z\|_r} = 0 \quad \Leftrightarrow \quad \begin{cases} \nabla_x T(x,y,z)^{\frac{1}{p-1}} & = \lambda^{\frac{1}{p-1}} x \\ \nabla_y T(x,\cdot,z)^{\frac{1}{q-1}} & = \lambda^{\frac{1}{q-1}} y \\ \nabla_z T(x,y,\cdot)^{\frac{1}{r-1}} & = \lambda^{\frac{1}{r-1}} z \end{cases}$$

Multihomogeneous mappings

$$g = (g_1, g_2) \colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+, \ \Theta \in \mathbb{R}^{2\times 2}, \ \alpha, \beta > 0$$

g is Θ-multihomogeneous $\Leftrightarrow$

$$g_1(\alpha x, \beta y) = \alpha^{\Theta_{11}} \beta^{\Theta_{12}} \, g_1(x, y)$$
$$g_2(\alpha x, \beta y) = \alpha^{\Theta_{21}} \beta^{\Theta_{22}} \, g_2(x, y)$$

Example

$$g_1(x, y) = (M^\top y)^{\frac{1}{p-1}}$$
$$g_2(x, y) = (Mx)^{\frac{1}{q-1}}$$

$$\Rightarrow \quad \Theta = \begin{pmatrix} 0 & \frac{1}{p-1} \\ \frac{1}{q-1} & 0 \end{pmatrix}$$

Multihomogeneous eigenvectors

$g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$, Θ-multihomogeneous,

(x, y) is eigenvector $\iff$ $x, y \neq 0$ and $\exists \lambda_1, \lambda_2 \geq 0$ s.t.

$$\begin{cases} g_1(x,y) = \lambda_1\, x \\ g_2(x,y) = \lambda_2\, y \end{cases}$$

$$\begin{aligned} g_1(x,y) &= \lambda_1\, x \\ g_2(x,y) &= \lambda_2\, y \end{aligned} \quad \iff \quad \begin{aligned} g_1(\alpha x, \beta y) &= \alpha^{\Theta_{11}-1}\beta^{\Theta_{12}}\lambda_1\, \alpha x \\ g_2(\alpha x, \beta y) &= \alpha^{\Theta_{21}}\beta^{\Theta_{22}-1}\lambda_2\, \beta y \end{aligned}$$

Multihomogeneous eigenvalues

$g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$, order-preserving, Θ-multihomogeneous,

$$\Theta \quad \text{irreducible} \qquad \Rightarrow \qquad \exists b \succ 0 \quad \text{s.t.} \quad \Theta b = \rho(\Theta) b$$

$$\begin{aligned} g_1(x,y) &= \lambda_1\, x \\ g_2(x,y) &= \lambda_2\, y \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} g_1(\alpha x, \beta y) &= (\alpha^{\Theta_{11}-1}\beta^{\Theta_{12}}\lambda_1)\,\alpha x \\ g_2(\alpha x, \beta y) &= (\alpha^{\Theta_{21}}\beta^{\Theta_{22}-1}\lambda_2)\,\beta y \end{aligned}$$

$$(\,\lambda_1^{b_1}\lambda_2^{b_2}\,)^{\rho(\Theta)} \quad = \quad (\alpha^{\Theta_{11}-1}\beta^{\Theta_{12}}\lambda_1)^{b_1}\,(\alpha^{\Theta_{21}}\beta^{\Theta_{22}-1}\lambda_2)^{b_2}$$

Multihomogeneous Hilbert metric

$g : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$, Θ-multihomogeneous, order-preserving

$$b \succ 0, \quad \Theta b = \rho(\Theta)\, b, \quad \|b\|_1 = 1$$

$$\mu_b\big(\,(x,y),\,(x',y')\,\big) = b_1\, d_H(x,x') + b_2\, d_H(y,y')$$

Lemma [G., Hein, Tudisco (2019)]

$$\mu_b\big(g(x,y),\, g(x',y')\big) \;\leq\; \rho(\Theta)\, \mu_b\big(\,(x,y),\,(x',y')\,\big)$$

Multihomogeneous spectral radius

$g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+,$ Θ-multihomogeneous, continuous, order-preserving

$$\rho(\Theta) = 1, \quad b \succ 0, \quad \Theta b = b, \quad \|b\|_1 = 1$$

$$\rho_b(g) \quad = \quad \lim_{k\to\infty} \left(\sup_{(x,y)\gneq 0} \frac{\|g_1^k(x,y)\|^{b_1}\, \|g_2^k(x,y)\|^{b_2}}{\|x\|^{b_1}\, \|y\|^{b_2}} \right)^{1/k}$$

$$g^k(x,y) = \big(g_1^k(x,y),\, g_2^k(x,y) \big)$$

Multihomogeneous Perron-Frobenius theorem I

$g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$, Θ-multihomogeneous, continuous, order-preserving

$$\rho(\Theta) = 1, \quad b \succ 0, \quad \Theta b = b, \quad \|b\|_1 = 1$$

Theorem [G., Tudisco, Hein (2019)]

$\sum_{k=0}^n g^k(x, y) \succ 0, \forall (x, y) \gneqq 0, \quad \Rightarrow \quad \exists (u, v) \succ 0$ s.t.

$g(u, v) = (\lambda_1 u, \lambda_2 v)$, $\quad \lambda_1^{b_1} \lambda_2^{b_2} = \rho_b(g) = \max\{\lambda_1^{b_1}\lambda_2^{b_2} : (\lambda_1, \lambda_2)$ eigenvalues$\}$

$\sum_{k=0}^n Dg(u, v)^k(x, y) \succ 0, \forall (x, y) \gneqq 0 \quad \Rightarrow \quad (u, v)$ unique in $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$

Multihomogeneous Perron-Frobenius theorem II

$g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$, Θ-multihomogeneous, continuous, order-preserving

$$\rho(\Theta) = 1, \quad b \succ 0, \quad \Theta b = b, \quad \|b\|_1 = 1$$

$$g(u, v) = (\lambda_1 u, \lambda_2 v), \quad (u, v) \succ 0, \quad \|u\| = \|v\| = 1$$

Theorem [G., Hein, Tudisco (2019)]

$$Dg(u, v)^k (x, y) \succ 0, \quad \forall (x, y) \gneq 0,$$

$$\Rightarrow \qquad \lim_{k\to\infty} \left(\frac{x^{(k)}}{\|x^{(k)}\|}, \frac{y^{(k)}}{\|y^{(k)}\|} \right) = (u, v)$$

$$\forall (x^{(0)}, y^{(0)}) \succ 0, \quad (x^{(k)}, y^{(k)}) = g(x^{(k-1)}, y^{(k-1)})$$

Multihomogeneous Perron-Frobenius theorem III

$g : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^n_{++} \times \mathbb{R}^n_{+}$, Θ-multihomogeneous, continuous, order-preserving

$$b \succ 0, \quad \Theta b = \rho(\Theta) b, \quad \|b\|_1 = 1$$

Theorem [G., Hein, Tudisco (2019)]

$0 < \rho(\Theta) < 1 \quad \Rightarrow \quad$ Exists unique $(u, v) \succ 0$ s.t.

$$g(u, v) = (\lambda_1 u, \lambda_2 v) \qquad \|u\| = \|v\| = 1.$$

and $\quad \forall (x^{(0)}, y^{(0)}) \succ 0, \quad (x^{(k)}, y^{(k)}) = g(x^{(k-1)}, y^{(k-1)})$

$$\lim_{k\to\infty} \left(\frac{x^{(k)}}{\|x^{(k)}\|}, \frac{y^{(k)}}{\|y^{(k)}\|} \right) = (u, v), \qquad \mu_b\big((x^{(k)}, y^{(k)}), (u, v)\big) \leq \rho(\Theta)^k C$$

Modelling: $\|T\|_{p,q,r}$

$T \in \mathbb{R}_+^{n\times n\times n}$, $1 < p, q, r < \infty$

$$\|T\|_{p,q,r} \quad = \quad \max_{x,y,z\neq 0} \frac{T(x,y,z)}{\|x\|_p\|y\|_q\|z\|_r}, \qquad T(x,y,z) = \sum_{i,j,k=1}^{n} T_{ijk}x_i y_j z_k$$

$$\nabla \frac{T(x,y,z)}{\|x\|_p\|y\|_q\|z\|_r} = 0 \qquad \Leftrightarrow \qquad \begin{cases} \nabla_x T(x,y,z)^{\frac{1}{p-1}} & = \lambda_1 x \\ \nabla_y T(x,y,z)^{\frac{1}{q-1}} & = \lambda_2 y \\ \nabla_z T(x,y,z)^{\frac{1}{r-1}} & = \lambda_3 z \end{cases}$$

$$\Theta = \begin{pmatrix} 0 & \frac{1}{p-1} & \frac{1}{p-1} \\ \frac{1}{q-1} & 0 & \frac{1}{q-1} \\ \frac{1}{r-1} & \frac{1}{r-1} & 0 \end{pmatrix}$$

Application: $\|T\|_{p,q,r}$

$T \in \mathbb{R}_+^{n\times n\times n}$, $1 < p,q,r < \infty$

$$\|T\|_{p,q,r} \quad = \quad \max_{x,y,z\neq 0} \frac{T(x,y,z)}{\|x\|_p \|y\|_q \|z\|_r}, \qquad T(x,y,z) = \sum_{i,j,k=1}^{n} T_{ijk} x_i y_j z_k$$

$$\Theta = (\mathbb{1}\mathbb{1}^\top - I)\,\mathrm{diag}\left(\tfrac{1}{p-1}, \tfrac{1}{q-1}, \tfrac{1}{r-1}\right)$$

Corollary [G., Hein, Tudisco (2019)]

$\rho(\Theta) = 1$ and $D\nabla T(x,y,z) \in \mathbb{R}_+^{(n+n+n)\times(n+n+n)}$ primitive, $\forall (x,y,z) \gneq 0$

$\Rightarrow$ compute unique positive global maximizer

$\rho(\Theta) < 1$ and $\nabla T(1,\dots,1) \succ 0$

$\Rightarrow$ compute unique positive global maximizer

General case

$$\begin{cases} f_1(x^{(1)}, x^{(2)}, \dots, x^{(m)}) & = \lambda_1 \, x^{(1)} \\ f_2(x^{(1)}, x^{(2)}, \dots, x^{(m)}) & = \lambda_2 \, x^{(2)} \\ & \vdots \\ f_m(x^{(1)}, x^{(2)}, \dots, x^{(m)}) & = \lambda_m \, x^{(m)} \end{cases} \quad \Rightarrow \quad \Theta \in \mathbb{R}^{m \times m}$$

$$\|T\|_{p_1,\dots,p_m} = \max_{x^{(1)},\dots,x^{(m)} \neq 0} \frac{T(x^{(1)}, \dots, x^{(m)})}{\|x^{(1)}\|_{p_1} \cdots \|x^{(m)}\|_{p_m}}$$

General cones: Polyhedral, Lorentz, PSD matrices, $\mathbb{R}_+^{n \times n \times n}$

Thank you

https://ftudisco.github.io/siam-nonlinear-pf-tutorial/